Uniqueness of solution for elliptic problems with non-linear boundary conditions

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ABSTRACT: In this paper we present results of uniqueness for an elliptic problem with nonlinear boundary conditions.
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1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega := \Gamma_0 \cup \Gamma_1$ where Γ_0 and Γ_1 are open and closed sets and $\Gamma_1 \cap \Gamma_0 = \emptyset$. Consider

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ Bu = h(x,u) & \text{on } \Gamma_1, \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $h : \Gamma_1 \times \mathbb{R} \mapsto \mathbb{R}$ are regular functions,

$$Bu := \frac{\partial u}{\partial n} + b(x)u,$$

with n the outward normal direction to $\partial\Omega$, $b \in C^{1,\alpha}(\Gamma_1)$, $\alpha \in (0,1)$ and $\mathcal{L}$ is a second order uniformly elliptic operator of the form

$$\mathcal{L}u := -\sum_{i,j=1}^{N} a_{ij} D_{ij} u + \sum_{i=1}^{N} b_i D_i u + c(x)u$$

with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$, $b_i, c \in C^{\alpha}(\overline{\Omega})$, $a_{ij} = a_{ji}$ and $0 < \alpha < 1$.

We present three results of uniqueness of solution of (1). Before stating our main results, we need some notations. Denote by

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_0, \\ Bu & \text{on } \Gamma_1, \end{cases}$$

and by $\sigma_1[\mathcal{L}, B]$ the principal eigenvalue (see for example Amann [3] and Cano-Casanova and López-Gómez [7]) of the problem

$$\begin{cases} \mathcal{L}u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Our first result is:

Theorem 1 *Assume $\sigma_1[\mathcal{L}, \mathcal{B}] > 0$. If $u \mapsto f(x,u), h(x,u)$ are non-increasing, then there exists at most a solution of* (1).

This result is well-known when $c \geq 0$, $\Gamma_0 = \emptyset$ and $Bu := \beta_0 u + \delta \dfrac{\partial u}{\partial \beta}$ (β an outward pointing vector field on $\partial\Omega$) with either $\beta_0 = 1$ and $\delta = 0$ (Dirichlet case), $\delta = 1$ and $\beta_0 = 0$ (Neumann case) or $\delta = 1$ and $\beta_0 > 0$ (regular oblique derivative boundary operator), see Amann [1] and Serrin [13]. In this paper, we generalize the result allowing more general boundary conditions and b and c could change sign.

Now, consider the uniqueness of positive solution. We denote by

$$P := \{u \in C^1(\overline{\Omega}) : \mathcal{B}u = 0,\ u(x) \geq 0,\ u \neq 0 \quad \text{in } \Omega \cup \Gamma_1\},$$

whose interior is

$$int(P) = \{u \in P : u(x) > 0 \quad \text{for all } x \in \Omega \cup \Gamma_1,\ \partial u/\partial n < 0 \quad \text{on } \Gamma_0.\}$$

We say that u is a positive solution of (1) if $u \in P$, and that is strictly positive, and we write $u \gg 0$, if $u \in int(P)$. Our second result is:

Theorem 2 *Assume that*

$$u \mapsto \frac{f(x,u)}{u},\ \frac{h(x,u)}{u}, \quad \textit{are non-increasing in } (0,\infty), \tag{3}$$

with one of them decreasing. Then there exists at most a positive solution of (1).

This result generalizes the classical one under homogeneous Dirichlet boundary condition (although the proof can be extended easily to the Robin case), which assures that if for a. e. $x \in \Omega$ the map

$$u \mapsto \frac{f(x,u)}{u} \quad \text{is decreasing in } (0,\infty) \tag{4}$$

then, there exists at most a positive solution of (1), see for instance Brezis and Kamin [4], Brezis and Oswald [5] and Hess [10].

Under condition (3), Theorem 2 was proved by Pao [12], Theorem 4.6.3, when $\Gamma_0 = \emptyset$, $\mathcal{L}$ self-adjoint, $b \geq 0$ and assuming the existence of a ordered pair of sub-supersolution, see also Umezu [14] for a related result under the more restrictive condition f/g decreasing.

Finally, in Delgado and Suárez [8] an extension to the classical result under condition (4) was given, and it was shown that the result complements and improves the above one. In this paper we generalize the result to nonlinear boundary conditions.

Theorem 3 *Assume $\sigma_1[\mathcal{L},\mathcal{B}] > 0$ and there exists $g \in C^1(0,+\infty) \cap C^0([0,+\infty))$, $g(t) > 0$ for $t > 0$ and g' non-increasing, such that*

$$u \mapsto \frac{f(x,u)}{g(u)}, \ \frac{h(x,u)}{g(u)} \quad \text{are non-increasing in } (0,\infty). \tag{5}$$

If:

1. $$\int_0^r \frac{1}{g(t)}dt < \infty, \qquad \text{for some } r > 0, \tag{6}$$

   then there exists at most a positive solution of (1).

2. $$\lim_{s\to 0} \frac{s}{g(s)} = 0, \tag{7}$$

   then there exists at most a strictly positive solution of (1).

In the following section we prove Theorems 1 and 3. For that, we use appropriate changes of variables. We also show that the condition $\sigma_1[\mathcal{L},\mathcal{B}] > 0$ is optimal in Theorem 1. In the third section we prove Theorem 2. Finally, in the last section we prove the existence and uniqueness of positive solution of the linear problem associated to (1).

2. PROOF OF THEOREMS 1 AND 3

2.1. AN IMPORTANT CHANGE OF VARIABLE

Since $\sigma_1[\mathcal{L},\mathcal{B}] > 0$, there exists $e \gg 0$ (in fact $e(x) > 0$ for all $x \in \overline{\Omega}$) the unique solution of (see Section 4)

$$\begin{cases} \mathcal{L}e = 0 & \text{in } \Omega, \\ e = 1 & \text{on } \Gamma_0, \\ Be = 0 & \text{on } \Gamma_1. \end{cases} \tag{8}$$

We make the change of variable

$$u := ev,$$

which transforms (1) into

$$\begin{cases} \mathcal{L}_1 v = f_1(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial n} = h_1(x,v) & \text{on } \Gamma_1, \end{cases} \tag{9}$$

where

$$\mathcal{L}_1 v := -\sum_{i,j=1}^{N} a_{ij} D_{ij} v + \sum_{i=1}^{N} b_i^1 D_i v, \tag{10}$$

with

$$b_i^1 := \left(b_i - \frac{2}{e} \sum_{j=1}^{N} a_{ij} D_j e \right),$$

and

$$f_1(x,v) := \frac{f(x,ev)}{e}, \quad h_1(x,v) := \frac{h(x,ev)}{e}. \tag{11}$$

Moreover, under the same change of variable, the problem (2) transforms into

$$\begin{cases} \mathcal{L}_1 v = \lambda v & \text{in } \Omega, \\ \mathcal{N} v = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where

$$\mathcal{N} v := \begin{cases} v & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial n} & \text{on } \Gamma_1, \end{cases}$$

and so,

$$\sigma_1[\mathcal{L}_1, \mathcal{N}] = \sigma_1[\mathcal{L}, \mathcal{B}] > 0.$$

From now on, we focus our attention on problem (9).

2.2. PROOF OF THEOREM 1

First observe that if f and h satisfy the conditions of Theorem 1, then the functions f_1 and h_1 defined in (11) are also non-increasing in v.

Take $v_1 \neq v_2$ two solutions of (9) and denote by

$$\Omega_1 := \{x \in \overline{\Omega} : v_1(x) > v_2(x)\}, \quad \text{and} \quad w := v_1 - v_2.$$

Then,

$$\begin{cases} \mathcal{L}_1 w \leq 0 & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \\ \dfrac{\partial w}{\partial n} \leq 0 & \text{on } \partial\Omega_1 \cap \Gamma_1. \end{cases} \tag{13}$$

It follows by the maximum principle (see for instance Theorem 3.5 in Gilbarg and Trudinger [9]) that the maximum of w has to be attained on $\partial\Omega_1 \cap \Gamma_1$ and that in such point $\partial w/\partial n > 0$ (see Lemma 3.4 in Gilbarg and Trudinger [9]), which is a contradiction with $\partial w/\partial n \leq 0$. $\square$

Remark 4 *Theorem 1 is not true if* $\sigma_1[\mathcal{L},\mathcal{B}]<0$. *Indeed, consider the logistic equation*

$$\begin{cases} \mathcal{L}u=\lambda u-u^p & en\ \Omega, \\ \mathcal{B}u=0 & en\ \partial\Omega, \end{cases} \tag{14}$$

where $p>1$ *and* $\lambda\in\mathbb{R}$. *It is well-known (see Cano-Casanova [6]) that* (14) *possesses the trivial solution* $u\equiv 0$ *for all* $\lambda\in\mathbb{R}$ *and for* $\lambda>\sigma_1[\mathcal{L},\mathcal{B}]$ *possesses another positive solution. Observe that* (14) *can be written as*

$$(\mathcal{L}-\lambda)u=-u^p.$$

In this case $f(x,u)=-u^p$ *is decreasing and* $\sigma_1[\mathcal{L}-\lambda,\mathcal{B}]=\sigma_1[\mathcal{L},\mathcal{B}]-\lambda<0$ *if* $\lambda>\sigma_1[\mathcal{L},\mathcal{B}]$.

2.3. PROOF OF THEOREM 3

Observe again that if f and g satisfy conditions of Theorem 3, then there exists a function $g_1\in C^1(0,+\infty)\cap C^0([0,+\infty))$ such that f_1, h_1 and g_1 satisfy also conditions of Theorem 3.

1. Assume (6) and let v a positive solution of (9). The change of variable

$$w:=\int_0^v \frac{1}{g_1(t)}dt \tag{15}$$

transforms (9) into

$$\begin{cases} \mathcal{L}_1 w=\dfrac{f_1(x,k(w))}{g_1(k(w))}+g_1'(k(w))\displaystyle\sum_{i,j=1}^N a_{ij}D_iwD_jw & \text{in } \Omega, \\ w=0 & \text{on } \Gamma_0, \\ \dfrac{\partial w}{\partial n}=\dfrac{h_1(x,k(w))}{g_1(k(w))} & \text{on } \Gamma_1, \end{cases} \tag{16}$$

where

$$v=k(w), \tag{17}$$

and k satisfies, from (15), that $k'(t)=g_1(k(t))$.

Assume that there exist two positive solutions $v_1\neq v_2$ of (9). Denote

$$\Omega_1:=\{x\in\overline{\Omega}: v_1(x)>v_2(x)\} \quad \text{and} \quad \Phi:=w_1-w_2,$$

where $v_i=k(w_i)$ $i=1,2$. Observe that $\Phi>0$ in Ω_1 thanks to the monotony of k. We have that in Ω_1

$$\begin{aligned} \mathcal{L}_1\Phi=&\left(\frac{f_1(x,k(w_1))}{g_1(k(w_1))}-\frac{f_1(x,k(w_2))}{g_1(k(w_2))}\right)+ \\ &+\left(g_1'(k(w_1))\sum_{i,j=1}^N a_{ij}D_iw_1D_jw_1-g_1'(k(w_2))\sum_{i,j=1}^N a_{ij}D_iw_2D_jw_2\right), \end{aligned} \tag{18}$$

$$\Phi = 0 \qquad \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \tag{19}$$

and

$$\frac{\partial \Phi}{\partial n} = \frac{h_1(x, k(w_1))}{g_1(k(w_1))} - \frac{h_1(x, k(w_2))}{g_1(k(w_2))} \qquad \text{on } \partial\Omega_1 \cap \Gamma_1. \tag{20}$$

Observe that,

$$g_1'(k(w_1)) \sum_{i,j=1}^{N} a_{ij} D_i w_1 D_j w_1 - g_1'(k(w_2)) \sum_{i,j=1}^{N} a_{ij} D_i w_2 D_j w_2 =$$

$$g_1'(k(w_1)) \sum_{i,j=1}^{N} a_{ij} D_j (w_1 + w_2) D_i \Phi + [g_1'(k(w_1)) - g_1'(k(w_2))] \sum_{i,j=1}^{N} a_{ij} D_i w_2 D_j w_2.$$

Moreover, thanks to that g_1' is non-increasing and that $\mathcal{L}$ is uniformly elliptic, it follows that

$$[g_1'(k(w_1)) - g_1'(k(w_2))] \sum_{i,j=1}^{N} a_{ij} D_i w_2 D_j w_2 \leq 0.$$

Thus, we get from (18) – (20) that

$$\begin{cases} \mathcal{L}_2 \Phi \leq 0 & \text{in } \Omega_1, \\ \Phi = 0 & \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0), \\ \dfrac{\partial \Phi}{\partial n} \leq 0 & \text{on } \partial\Omega_1 \cap \Gamma_1, \end{cases} \tag{21}$$

where

$$\mathcal{L}_2 \Phi := -\sum_{i,j=1}^{N} a_{ij} D_{ij} \Phi + \sum_{i=1}^{N} \left(b_i^1 - g'(k(w_1)) \sum_{j=1}^{N} a_{ij} D_j (w_1 + w_2) \right) D_i \Phi.$$

It suffices to apply again the strong maximum principle.

2. Assume now (7) and that there exist two strictly positive solutions $v_1 \neq v_2$ of (9) with $v_i \in int(P)$, $i = 1, 2$. Let $\Omega_1 := \{x \in \overline{\Omega} : v_1(x) > v_2(x)\}$. We define now for $x \in \Omega_1$

$$\Phi(x) := \int_{v_2(x)}^{v_1(x)} \frac{1}{g_1(t)} dt.$$

First, observe that

$$\Phi = 0 \quad \text{on } \partial\Omega_1 \cap (\Omega \cup \Gamma_0).$$

Indeed, for $x \in \partial\Omega_1 \cap \Omega$ it is clear that $\Phi(x) = 0$. For $x \in \Omega_1$ we have that for some $\xi(x)$ with $v_2(x) \leq \xi(x) \leq v_1(x)$

$$\Phi(x) = \frac{v_1(x) - v_2(x)}{g_1(\xi(x))} \leq \frac{C dist(x)}{g_1(\xi(x))} \to 0,$$

as $dist(x) \to 0$, where $dist(x) = dist(x, \partial\Omega)$ thanks to (7). Hence $\Phi = 0$ on $\partial\Omega_1 \cap \Gamma_0$. Now, the proof follows as the case a) (see Proposition 2.2 in Delgado and Suárez [8]). $\square$

3. PROOF OF THEOREM 2

First, observe that if u is a positive solution of (1) then u is strictly positive. Indeed, since $u \leq \|u\|_\infty$, it follows that

$$\frac{f(x,u)}{u} \geq \frac{f(x,\|u\|_\infty)}{\|u\|_\infty} := -K_1, \qquad \frac{h(x,u)}{u} \geq \frac{h(x,\|u\|_\infty)}{\|u\|_\infty} := -K_2.$$

Take $M > \max\{0, K_1, K_2, -\sigma_1[\mathcal{L},\mathcal{B}]\}$. Then, (1) is equivalent to

$$\mathcal{L}u + Mu = f(x,u) + Mu > 0 \quad \text{in } \Omega, \qquad \mathcal{B}u + Mu = h(x,u) + Mu > 0 \quad \text{in } \partial\Omega.$$

Moreover, thanks to the monotonicity properties of the principal eigenvalue (see Proposition 3.5 in Cano-Casanova and López-Gómez [7]), we get that

$$\sigma_1[\mathcal{L}+M,\mathcal{B}+M] > \sigma_1[\mathcal{L}+M,\mathcal{B}] = M + \sigma_1[\mathcal{L},\mathcal{B}] > 0,$$

and so, the strong maximum principle (for instance Theorem 2.1 in Cano-Casanova and López-Gómez [7]) concludes that $u \gg 0$.

Take two positive solutions $u_1 \neq u_2$ of (1) and define

$$w := u_1 - u_2.$$

Since u_1 is a strictly positive solution of (1), then

$$\sigma_1[\mathcal{L} - \frac{f(x,u_1)}{u_1}, \mathcal{B} - \frac{h(x,u_1)}{u_1}] = 0. \tag{22}$$

It is not hard to show that

$$\mathcal{L}w - F(x)w = 0 \quad \text{in } \Omega, \qquad \mathcal{B}w - H(x)w = 0 \quad \text{on } \partial\Omega, \tag{23}$$

where

$$F(x) := \begin{cases} \dfrac{f(x,u_1) - f(x,u_2)}{u_1 - u_2} & u_1 \neq u_2, \\ D_2 f(x,u_1) & u_1 = u_2, \end{cases} \qquad H(x) := \begin{cases} \dfrac{h(x,u_1) - h(x,u_2)}{u_1 - u_2} & u_1 \neq u_2, \\ D_2 h(x,u_1) & u_1 = u_2. \end{cases}$$

Hence, from (23) it follows that 0 is an eigenvalue of the operator $\mathcal{L} - F$ under homogeneous boundary condition $\mathcal{B} - H$, that is

$$0 = \sigma_j[\mathcal{L} - F, \mathcal{B} - H], \quad \text{for some } j \geq 1.$$

On the other hand, thanks to (3), it follows that

$$F(x) \leq \frac{f(x,u_1)}{u_1} \quad \text{and} \quad H(x) \leq \frac{h(x,u_1)}{u_1},$$

and one of the inequalities strict. Thus,

$$0 = Re(\sigma_j[\mathcal{L} - F, \mathcal{B} - H]) \geq \sigma_1[\mathcal{L} - F, \mathcal{B} - H] > \sigma_1[\mathcal{L} - \frac{f(x,u_1)}{u_1}, \mathcal{B} - \frac{h(x,u_1)}{u_1}] = 0,$$

a contradiction. $\square$

Remark 5 *If instead of* (3)*, we assume that both maps are non-decreasing, we can conclude that if* u_1 *and* u_2 *are ordered, then* $u_1 = u_2$*.*

4. THE LINEAR PROBLEM

In this section we give a result of existence and uniqueness of a linear problem.

Proposition 6 *Assume that* $\sigma_1[\mathcal{L}, \mathcal{B}] > 0$*,* $(f, g, h) \in C^\alpha(\overline{\Omega}) \times C^{1,\alpha}(\Gamma_0) \times C^{1,\alpha}(\Gamma_1)$*, such that* $f, g, h \geq 0$ *and some of the inequalities strict. Then, there exists a unique strictly positive solution of the linear problem*

$$\begin{cases} \mathcal{L}u = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \Gamma_0, \\ Bu = h(x) & \text{on } \Gamma_1. \end{cases} \tag{24}$$

Proof: Since Ω is smooth, there exists (see López-Gómez [11], Proposition 3.4) $\psi \in C^{2,\alpha}(\overline{\Omega})$ and a constant $\gamma > 0$ such that

$$\frac{\partial \psi}{\partial n} \geq \gamma > 0 \quad \text{on } \Gamma_1. \tag{25}$$

We make the following change of variable

$$u := e^{M\psi} v. \tag{26}$$

Under this change, (24) transforms into

$$\begin{cases} \mathcal{L}_M v = f_M(x) & \text{in } \Omega, \\ v = g_M(x) & \text{on } \Gamma_0, \\ B_M v = h_M(x) & \text{on } \Gamma_1, \end{cases} \tag{27}$$

where

$$f_M = f e^{-M\psi}, \quad g_M = g e^{-M\psi}, \quad h_M = h e^{-M\psi},$$

$$\mathcal{L}_M v := -\sum_{i,j=1}^N a_{ij} D_{ij} v + \sum_{i=1}^N b_i^M D_i v + c_M(x) v, \qquad B_M v := \frac{\partial v}{\partial n} + b_M(x) v,$$

and

$$b_i^M := \left(b_i - 2M \sum_{j=1}^N a_{ij} D_j \psi \right), \qquad b_M := (b(x) + M \frac{\partial \psi}{\partial n}),$$

$$c_M := c(x) + M \sum_{i=1}^N b_i D_i \psi - M \sum_{i,j=1}^N a_{ij} D_{ij} \psi - M^2 \sum_{i,j=1}^N a_{ij} D_i \psi D_j \psi.$$

On the other hand, (2) transforms into

$$\begin{cases} \mathcal{L}_M v = \lambda v & \text{in } \Omega, \\ \mathcal{B}_M v = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

and so,

$$\sigma_1[\mathcal{L}, \mathcal{B}] = \sigma_1[\mathcal{L}_M, \mathcal{B}_M].$$

Thanks to (25), we can take $M > 0$ large enough such that

$$b_M \geq 0.$$

Now, we focus our attention on solving (27). Take a regular function $K(x)$ such that

$$K(x) > \max\{c_M(x), 0\}$$

and consider the unique positive solution (which exists because $b_M, K \geq 0$, see Gilbarg and Trudinger [9], Theorem 6.1) of

$$\begin{cases} (\mathcal{L}_0 + K(x))w = f_M(x) & \text{in } \Omega, \\ w = g_M(x) & \text{on } \Gamma_0, \\ B_M w = h_M(x) & \text{on } \Gamma_1, \end{cases} \tag{29}$$

where

$$\mathcal{L}_0 w := -\sum_{i,j=1}^{N} a_{ij} D_{ij} w + \sum_{i=1}^{N} b_i^M D_i w.$$

Now, it is evident that a solution v of (27) can be written as $v = z + w$ with z solution of

$$\begin{cases} \mathcal{L}_M z = f_1(x) := [K(x) - c_M(x)]w > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma_0, \\ B_M z = 0 & \text{on } \Gamma_1. \end{cases} \tag{30}$$

So, it remains to show that (30) possesses a unique positive solution, for that we are going to use the classical Riesz Theory. Observe that $\mathcal{L}_M z = f_1(x)$ is equivalent to

$$(\mathcal{L}_M + R)z - Rz = f_1(x) \Longleftrightarrow \frac{1}{R}z - (\mathcal{L}_M + R)^{-1}z = f_2(x) := (\mathcal{L}_M + R)^{-1}(\frac{f_1(x)}{R}) \geq 0,$$

where R is a positive constant sufficiently large so that $\sigma_1[\mathcal{L}_M + R, \mathcal{B}_M] > 0$, and so there exists the inverse of $\mathcal{L}_M + R$ under homogeneous boundary condition $\mathcal{B}_M$. Denoting $r(T)$ the spectral radius of a linear operator T, we get that

$$\frac{1}{R} > r((\mathcal{L}_M + R)^{-1}) = \frac{1}{\sigma_1[\mathcal{L}_M + R, \mathcal{B}_M]} = \frac{1}{\sigma_1[\mathcal{L}_M, \mathcal{B}_M] + R},$$

thanks to $\sigma_1[\mathcal{L}_M, \mathcal{B}_M] > 0$. It now suffices to apply Theorem 3.2 of Amann [2] and the result concludes. □

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